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Laminar film flow on a cylindrical surface

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The paper deals with steady laminar film flow which is set up at the cylindrical surface of an idealized horizontal 'road' when homogeneous 'rain' is falling onto the road in a vertical downward direction. It is shown that a particular solution of the Navier–Stokes equations is possible for which the depth of the liquid film is constant. In that case the Navier–Stokes equations reduce to the equations governing plane stagnation-point flow. However, the boundary conditions differ from those for the classical stagnation-point problem. Solutions for nearly inviscid flow and predominantly viscous flow are derived analytically. In particular, simple formulae for the depth of the film are found in both cases. Finally, the importance of the particular solution as a member of a whole class of solutions is discussed on the basis of a momentum integral approximation.

1. Introduction

In connexion with the investigation of the flow of rain-water over a curved road surface by Schleicher (1975) the following idealized hydrodynamic problem arose (figure 1): over a rigid cylindrical surface ('road surface') with constant radius of curvature R and a horizontal axis, a liquid film of thickness h is flowing under the action of gravity. The flow is assumed laminar, two-dimensional and steady. The amount of liquid flowing in the film is continuously supplied by 'rain', falling with constant velocity V in a vertical downward direction onto the surface of the film. The 'rain' is idealized as a homogeneous medium with density $\epsilon\rho$, where ρ is the density of the liquid in the film and ϵ is a number between 0 and 1: $0 < \epsilon < 1$. In applications to problems of real rain flow, ϵ is of the order of 10^{-6}; hence the assumption $\epsilon \ll 1$ is justified in such cases. However, to keep the theory as general as possible, and with an eye on possible applications to artificial sprinkling for which ϵ might be much larger, this assumption will not be made here.

The main part of this paper is devoted to the derivation and discussion of a particular solution of the hydrodynamic equations for which the film thickness has the same value everywhere. In § 2 the Navier–Stokes equations and boundary conditions are simplified by assuming constant film thickness. The resulting equations are those for plane stagnation-point flow. However, the boundary conditions to be satisfied at the film surface induce complications which are absent in classical stagnation-point flow. These boundary conditions are derived from a momentum balance at the film surface. The film surface, in a certain

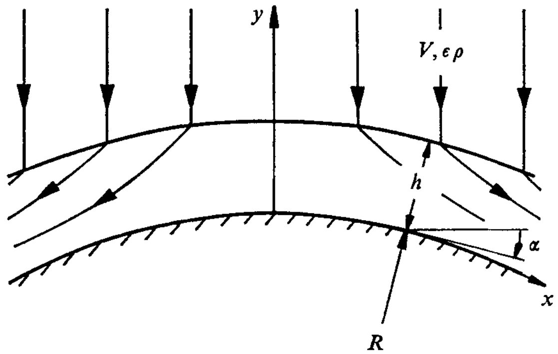

FIGURE 1. Notation for film flow with constant depth h.

sense, is analogous to an oblique shock wave in gasdynamics: the density of the medium on crossing the surface jumps from $\epsilon\rho$ to ρ. The concomitant change in normal stress (essentially pressure) causes an augmentation of the 'effective gravity' driving the flow. Therefore, in all results, instead of the acceleration due to gravity g, an 'effective' acceleration due to gravity g^* appears.

In §3 the equations for the liquid film are solved, for nearly inviscid flow, by matched asymptotic expansions. The flow field consists of a boundary layer (the classical stagnation-point boundary layer to a first approximation) near the wall and an inviscid flow near the surface of the film. The inviscid flow is rotational. The rotation is induced by the passage of the liquid through the free surface (see also Becker 1975).

For predominantly viscous flow a solution is found in §4 by a regular power-series expansion. The results obtained are of immediate relevance for rain flow on a road, because such flows are within the domain of validity of these results.

In §5, an approximate 'hydraulic' method, based on the momentum integral, is used to clarify the significance of the solution discussed so far, as a particular solution within a whole class of solutions. The discussion indicates that the particular solution is realized in most cases of practical importance for rain-water flow.

Throughout §§2–5 the thickness of the film is assumed so small that certain curvature terms in the underlying equations may be neglected. The justification of this neglect is discussed in §6 and a numerical example is given there.

2. Statement of assumptions and derivation of equations

An x, y co-ordinate system is introduced, as shown in figure 1, with u and v denoting the velocity components in the x and y directions, i.e. parallel and normal to the bounding wall. Assuming the thickness of the film h to be small compared with the radius of wall curvature R, we neglect curvature terms of the order $h/R \ll 1$ in the Navier–Stokes equations. An assessment of the limitations imposed by this neglect is presented in §6. The Navier–Stokes equations then assume the form

$$u_x + v_y = 0, \tag{2.1}$$

$$uu_x + vu_y = -\rho^{-1}p_x + g\sin\alpha + \nu\Delta u, \tag{2.2}$$

$$uv_x + vv_y = -\rho^{-1}p_y - g\cos\alpha + \nu\Delta v. \tag{2.3}$$

We now define
$$p^* = p - \rho g\int_0^x \sin\alpha\, dx + \rho g y\cos\alpha. \tag{2.4}$$

Taking account of $d\alpha/dx = 1/R$, we obtain from (2.4)

$$p^*_x = p_x - \rho g\sin\alpha(1 + y/R) \tag{2.5}$$

and
$$p^*_y = p_y + \rho g\cos\alpha. \tag{2.6}$$

In view of the assumption $h/R \ll 1$, the term y/R is now omitted from (2.5). As a consequence (2.2) and (2.3) may be written as

$$uu_x + vu_y = -\rho^{-1}p^*_x + \nu\Delta u \tag{2.7}$$

and
$$uv_x + vv_y = -\rho^{-1}p^*_y + \nu\Delta v. \tag{2.8}$$

In addition to these equations and (2.1) the flow has to satisfy boundary conditions at the rigid wall and at the free surface of the film. At the wall the no-slip condition applies:

$$u = v = 0 \quad \text{for} \quad y = 0. \tag{2.9}$$

The conditions to be satisfied at the surface of the film are derived from the balance of momentum at the surface (see also Becker 1975; Böhme & Becker 1972). As we shall restrict the following derivations to a particular solution for which the thickness h is independent of x, the slope of the film surface is the same as that of the wall, and is given by the angle α. Since the mass flow rate of rain perpendicular to a unit area of the film surface is $\epsilon\rho V\cos\alpha$, and since the tangential velocity jumps from $V\sin\alpha$ just above to $u(h)$ just below the surface, the tangential stress component jumps by the amount

$$\tau = \epsilon\rho V\cos\alpha(V\sin\alpha - u(h)). \tag{2.10}$$

Because the region above the surface is free of stress, the expression (2.10) gives the tangential stress immediately below the surface of the film. Likewise, the balance of momentum in the direction normal to the surface yields for the normal stress immediately below the surface, provided the pressure level above the surface is chosen as zero,

$$\sigma = -\epsilon\rho(1-\epsilon)\, V^2\cos^2\alpha. \tag{2.11}$$

Now, for a Newtonian fluid,
$$\tau = \nu\rho(u_y + v_x), \tag{2.12}$$

and
$$\sigma = -p + 2\nu\rho v_y, \tag{2.13}$$

where p denotes pressure. For the particular solution, to be derived presently, it so happens that v depends on y only [cf. equation (2.16)]. Therefore, combination of (2.12) with (2.10) leads to the following boundary condition:

$$\nu u_y = \epsilon V\cos\alpha(V\sin\alpha - u) \quad \text{for} \quad y = h. \tag{2.14}$$

From (2.13) one infers that

$$\sigma_x = -p_x \quad \text{for} \quad y = h. \tag{2.15}$$

For the solution of (2.7) and (2.8), subject to the boundary conditions (2.9) and (2.14), we now assume

$$u = xf'(y), \quad v = -f(y), \tag{2.16}$$

$$p^* = -\tfrac{1}{2}\rho b^2(x^2+F(y)), \tag{2.17}$$

with constant b^2. We note that (2.16) and (2.17) is the ansatz for plane stagnation-point flow (Schlichting 1965). By (2.16) the equation of continuity (2.1) is satisfied. The determination of the function $f(y)$, together with the thickness h of the film, is the main task of the subsequent calculations.

The constant b^2 in (2.17) can be derived from (2.11). On the one hand, differentiation of (2.11) and use of (2.15) yields

$$-\sigma_x = p_x = -2\rho\epsilon(1-\epsilon)R^{-1}V^2\sin\alpha\cos\alpha \quad \text{for} \quad y = h. \tag{2.18}$$

On the other hand, from (2.17) and (2.5) one obtains

$$p_x^* = p_x - \rho g\sin\alpha = -\rho b^2 x. \tag{2.19}$$

Combination of (2.18) and (2.19) leads to

$$b^2 x = g\sin\alpha(1+2\epsilon(1-\epsilon)\,V^2\cos\alpha/gR). \tag{2.20}$$

This relation is satisfied with constant b^2 if the values of α are restricted by the assumption that $|\alpha| \ll 1$, such that $\sin\alpha \approx x/R$ and $\cos\alpha \approx 1$. Then

$$b^2 = g^*/R, \tag{2.21}$$

where an 'effective' constant of gravity g^* has been defined as

$$g^* = g + 2\epsilon(1-\epsilon)\,V^2/R. \tag{2.22}$$

Insertion of (2.16) and (2.17) into (2.7) and (2.8) and use of (2.21) yields

$$f'^2 - ff'' = \nu f''' + g^*/R, \tag{2.23}$$

and

$$ff' + \nu f'' = F'g^*/(2R). \tag{2.24}$$

The boundary conditions (2.9) are transformed into

$$f(0) = f'(0) = 0. \tag{2.25}$$

The boundary condition (2.14), with $\cos\alpha = 1$ and $\sin\alpha = x/R$, is transformed into

$$\nu f''(h) = \epsilon V(V/R - f'(h)). \tag{2.26}$$

Finally, the balance of mass at the surface of the film yields another boundary condition, namely $v(h) = \epsilon V\cos\alpha$, which is equivalent to

$$f(h) = \epsilon V. \tag{2.27}$$

Equations (2.23)–(2.27) constitute the final formulation of the problem to be solved. The derivation of these equations is based on the assumptions

$$R = \text{constant}, \quad h/R \ll 1, \quad |\alpha| \ll 1. \tag{2.28}$$

The stipulation that h be independent of x is not an additional assumption, because it will be satisfied exactly by the solution to be found. Since the pressure distribution in the liquid film is of no particular concern, (2.24) will be omitted from now on.

It is convenient to cast the equations into a dimensionless form. For that purpose the following definitions are appropriate:

$$Re = (g^*R)^{\frac{1}{2}} R/\nu \quad \text{(Reynolds number)}, \tag{2.29}$$

$$M = V/(g^*R)^{\frac{1}{2}} \quad \text{(Froude number)}, \tag{2.30}$$

$$f(y) = (g^*R)^{\frac{1}{2}} w(\eta) \quad \text{with} \quad \eta = y/R. \tag{2.31}$$

Equations (2.23) and (2.25)–(2.27) thereby assume the following form:

$$w'^2 - ww'' = 1 + w'''/Re, \tag{2.32}$$

$$w(0) = w'(0) = 0, \tag{2.33}$$

$$w(\tilde{h}) = \epsilon M, \tag{2.34}$$

$$w''(\tilde{h}) = \epsilon M\, Re(M - w'(\tilde{h})). \tag{2.35}$$

Here, the abbreviation $\tilde{h} = h/R$ has been used. We note that

$$w'(\eta) = (R/g^*)^{\frac{1}{2}} f'(y) = u(x,y)/u_0(x). \tag{2.36}$$

Here,

$$u_0(x) = (g^*x^2/R)^{\frac{1}{2}} = (2g^*z)^{\frac{1}{2}} \tag{2.37}$$

is the velocity of free fall through a height $z = x^2/(2R)$, which is the vertical distance between the highest point of the wall at $x = 0$ and the wall at x.

It is to be noted that (2.32) is a *third*-order differential equation; the solution is subject to the *four* boundary conditions (2.33)–(2.35). This discrepancy between the order of the equation and the number of boundary conditions causes no inconsistency because the dimensionless film thickness $\tilde{h}$ is undetermined *a priori* and has to be calculated, as a kind of eigenvalue, together with the solution w.

3. Solution for nearly inviscid flow

For sufficiently large values of the Reynolds number the flow in the film may be divided into a viscous boundary-layer flow near the wall ('inner flow') and an inviscid flow near the film surface ('outer flow'). The outer solution of (2.32) is denoted by $w^{(a)}$, and the following asymptotic expansion for $Re \to \infty$ with ϵ and M fixed is postulated:

$$w^{(a)} = w_0 + Re^{-\frac{1}{2}} w_1 + Re^{-1} w_2 + \dots\ . \tag{3.1}$$

The film thickness $\tilde{h}$ is expanded in the same way:

$$\tilde{h} = \tilde{h}_0 + Re^{-\frac{1}{2}} \tilde{h}_1 + Re^{-1} \tilde{h}_2 + \dots\ . \tag{3.2}$$

Substituting (3.1) and (3.2) into (2.32) and the outer boundary conditions (2.34) and (2.35) and comparing like powers of $Re^{-\frac{1}{2}}$ we obtain

$$w_0'^2 - w_0 w_0'' = 1, \tag{3.3}$$

$$2w_0' w_1' - w_0 w_1'' - w_1 w_0'' = 0, \tag{3.4}$$

$$w_0(\tilde{h}_0) = \epsilon M, \tag{3.5}$$

$$\tilde{h}_1 w_0'(\tilde{h}_0) + w_1(\tilde{h}_0) = 0, \tag{3.6}$$

$$w_0'(\tilde{h}_0) = M, \tag{3.7}$$

$$\tilde{h}_1 w_0''(\tilde{h}_0) + w_1'(\tilde{h}_0) = 0. \tag{3.8}$$

The inner solution is assumed to have the form

$$w^{(i)} = Re^{-\frac{1}{2}}\phi(\zeta) = Re^{-\frac{1}{2}}(\phi_0 + Re^{-\frac{1}{2}}\phi_1 + \ldots). \tag{3.9}$$

Here ζ denotes a stretched co-ordinate:

$$\zeta = \eta Re^{\frac{1}{2}}. \tag{3.10}$$

Substitution of (3.9) into (2.32) yields

$$\phi_0'^2 - \phi_0\phi_0'' = 1 + \phi_0''', \tag{3.11}$$

$$\phi_1''' + \phi_1''\phi_0 - 2\phi_1'\phi_0' + \phi_1\phi_0'' = 0. \tag{3.12}$$

The inner boundary conditions (2.33) are equivalent to

$$\phi_n(0) = \phi_n'(0) = 0, \quad n = 0, 1, \ldots\ . \tag{3.13}$$

In addition to satisfying the outer and inner boundary conditions respectively, the outer and inner solution have to match in an overlapping region of common validity.

The zeroth-order outer solution $(w_0, \tilde{h}_0)$, which satisfies, in addition to the outer boundary conditions (3.5) and (3.7), the condition $w_0(0) = 0$, is given by

$$w_0 = \epsilon_1 \sin(\eta/\epsilon_1), \tag{3.14}$$

$$\tilde{h}_0 = \epsilon_1 \arccos M, \tag{3.15}$$

where $\epsilon_1 = \epsilon M(1-M^2)^{-\frac{1}{2}}$. This is the inviscid solution of the present film flow problem. The flow is rotational, except for $M = 1$, when $w_0' = 1$; the curl of the velocity field is proportional to w_0''. The rotation is induced at the free surface, where new liquid is continuously entrained. The solution (3.14) and (3.15) has already been found – in a different context – by Becker (1975). At the wall, $\eta = 0$, (3.14) gives $w_0' = 1$. From (2.36) it then follows that the velocity at the wall is $u(x, 0) = u_0(x) = (2g^*z)^{\frac{1}{2}}$. Hence, in inviscid flow, the liquid at the wall acquires the velocity of free fall through the height z, a result which was to be expected.

A solution of (3.11) and (3.13) which satisfies the additional boundary condition $\phi_0'(\infty) = 1$ is provided by the well-known classical solution for plane stagnation-point flow. The solutions w_0 and ϕ_0 match, as is shown by the following considerations. A 'matching' co-ordinate $s = \eta Re^{\frac{1}{4}}$ is defined such that $\eta = sRe^{-\frac{1}{4}}$ and $\zeta = sRe^{+\frac{1}{4}}$. At the present order of approximation the inner and outer solutions are given by

$$\begin{aligned} w^{(i)} &= Re^{-\frac{1}{2}}\phi_0(sRe^{\frac{1}{4}}) \\ &= Re^{-\frac{1}{2}}(sRe^{\frac{1}{4}} - \Delta + O(Re^{-\frac{1}{4}})), \end{aligned} \tag{3.16}$$

$$\begin{aligned} w^{(a)} &= w_0(sRe^{-\frac{1}{4}}) \\ &= sRe^{-\frac{1}{4}} + O(Re^{-\frac{3}{4}}). \end{aligned} \tag{3.17}$$

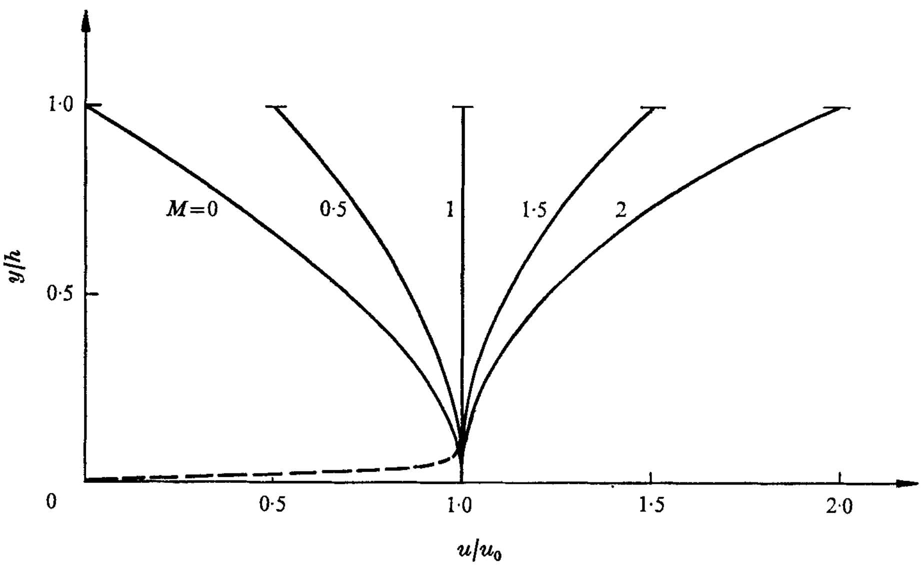

FIGURE 2. Inviscid velocity parallel to the bounding wall for different values of Froude number M. – – –, boundary-layer correction at the wall.

Here, the expansions for fixed s (in the common domain of validity) and $Re \rightarrow \infty$ have been indicated. The properties $w_0(0) = 0$, $w_0'(0) = 1$ and $\phi_0'(\infty) = 1$ have been used; the number

$$\Delta = \int_0^\infty (1-\phi_0')\,d\zeta$$

has the value 0·648 (see Schlichting 1965). It is easily seen that the first term of (3.16) matches the first term of (3.17).

Figure 2 shows the inviscid velocity distribution $w_0'(\eta)$. The co-ordinate η is normalized with the film thickness $\tilde{h}_0$, which is shown in figure 3. The correction of the inviscid velocity distribution due to the boundary layer at the wall is qualitatively indicated by the dotted line in figure 2.

It should be noted that for $M > 1$ the values of $(1-M^2)^{\frac{1}{2}}$, $\sin(\eta/\epsilon_1)$ and $\arccos M$ in (3.14) and (3.15) all become imaginary. Although the imaginary unit drops out of the results for w_0 and h_0, in that case it is more convenient to change in these results simultaneously $(1-M^2)^{\frac{1}{2}}$ into $(M^2-1)^{\frac{1}{2}}$ and the trigonometric functions into the corresponding hyperbolic functions. Incidentally, this remark (which also applies to (3.19) and (3.22)) shows that the inviscid solution is uniquely determined only for $M > 1$, because the function $Ar\cosh M$ in (3.15) has only one positive value for $M > 1$. The multivaluedness of $\arccos M$ in (3.15) indicates the existence, for $M < 1$, of an infinite number of branches of the solution! Each of these solution branches is characterized by a certain number of adjacent layers of fluid with opposite flow directions (up and down the slope of the wall) alternating in the y direction. The classical stagnation-point boundary layer matches *all* these solutions. However, in what follows it is always tacitly assumed that the simple one-layer solution with a unique flow

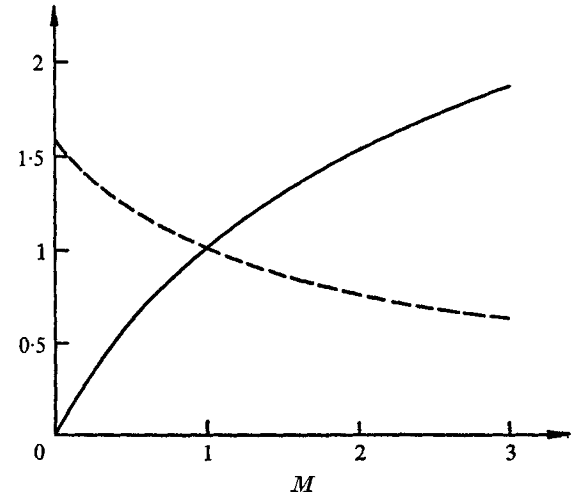

FIGURE 3. Film thickness for inviscid flow. ——, $\bar{h}_0/\epsilon$; – – –, $\bar{h}_0/\epsilon M$.

direction down the slope of the wall has been chosen; all diagrams are based on this assumption. This means that the lowest positive value of $\arccos M$ has to be selected in (3.15). Only this solution has a continuous extension to the case when $M > 1$ (see, for example, figure 3).

A two-term matching of the inner and outer solutions can be effected by proceeding to the outer solution of next order w_1. It is easily verified that

$$w_1 = A w_0' \tag{3.18}$$

with constant A is a solution of (3.4). The constant A is connected with the first-order correction $\bar{h}_1$ to the film thickness. By substituting (3.18) into the boundary conditions (3.6) and (3.8) one can see that both conditions are satisfied if

$$\bar{h}_1 = -A.$$

Therefore, the next approximation to the outer solution is

$$w^{(a)} = \epsilon_1 \sin(\eta/\epsilon_1) - \bar{h}_1 Re^{-\frac{1}{2}} \cos(\eta/\epsilon_1). \tag{3.19}$$

Introduction of the matching co-ordinate s and expansion for fixed s and $Re \to \infty$ leads to

$$w^{(a)} = s Re^{-\frac{1}{2}} - \bar{h}_1 Re^{-\frac{1}{2}} + O(Re^{-\frac{3}{2}}). \tag{3.20}$$

This matches the first two terms of (3.16) if

$$\bar{h}_1 = \Delta = 0{\cdot}648, \tag{3.21}$$

or

$$\bar{h} = \epsilon_1 \arccos M + 0{\cdot}648\, Re^{-\frac{1}{2}}. \tag{3.22}$$

The augmentation of the inviscid film thickness $\bar{h}_0$, given by the second term on the right-hand side of (3.22), is easily interpreted by noting that this term is the displacement thickness of the classical stagnation-point boundary layer, as determined by ϕ_0. This is a rather obvious result.

In order to improve on these results, one would have to solve (3.12) for ϕ_1. This is not possible without numerical integration. In view of the fact that the nearly inviscid flow is less important for applications, for example to the flow of rain-water on a road surface, than the predominantly viscous flow, we desist from such calculations and turn to an approximate solution for the case in which viscosity plays the dominant role.

4. Solution for predominantly viscous flow

For the following discussion it is convenient to introduce a new dimensionless parameter λ, defined by

$$\lambda = \epsilon M Re^{\frac{1}{2}} = \epsilon V(R/g^*\nu^2)^{\frac{1}{3}}. \tag{4.1}$$

The solution method explained in the preceding section certainly breaks down if the second term on the right-hand side of (3.22) is of the same order of magnitude or larger than the first term. Figure 3 shows that for $0 \leqslant M \lesssim 3$ the function $(1-M^2)^{-\frac{1}{2}}$ arc cos M is of order 1. Therefore, the condition that the two terms in (3.22) are of the same order of magnitude means, for moderate values of M, that the parameter λ is of order 1. Hence we conclude that the results of § 3 are valid only if $\lambda \gg 1$.

We now study the solution in the limit $\lambda \to 0$, for M fixed. In this case viscosity is dominant over the whole depth of the film. Equations (3.32)–(3.35) are transformed by putting

$$w = Re^{-\frac{1}{2}}\phi(\zeta), \quad \zeta = \eta Re^{\frac{1}{2}}. \tag{4.2}$$

The transformed equations are

$$\phi'^2 - \phi\phi'' = 1 + \phi''', \tag{4.3}$$

$$\phi(0) = \phi'(0) = 0, \tag{4.4}$$

$$\phi(\tilde{\zeta}) = \lambda, \tag{4.5}$$

$$\phi''(\tilde{\zeta}) = \lambda(M - \phi'(\tilde{\zeta})). \tag{4.6}$$

Here

$$\tilde{\zeta} = \tilde{h} Re^{\frac{1}{2}} = h Re^{\frac{1}{2}}/R. \tag{4.7}$$

We note that $dw/d\eta = d\phi/d\zeta$; therefore ϕ' is the dimensionless velocity distribution as explained in connexion with (2.36).

The solution for $\tilde{\zeta} = \tilde{\zeta}(\lambda, M)$ is now sought, for small λ, in form of an expansion in powers of λ. The problem posed by (4.3)–(4.6) has the property that for $\lambda \to 0$ the dimensionless film thickness $\tilde{\zeta}$ also tends to zero: $\tilde{\zeta} \to 0$. Therefore, the first terms of the power-series expansion of ϕ with respect to ζ are sufficient to find the first terms of the expansion of $\tilde{\zeta}$ as a function of λ:

$$\phi(\zeta) = \sum_{n=2} a_n \zeta^n. \tag{4.8}$$

By starting the expansion with the term $a_2\zeta^2$ we have already satisfied the conditions (4.4). Substitution of (4.8) into (4.3) and comparing like powers of ζ shows that all coefficients a_n, $n \geqslant 3$, can be related to the first coefficient $a_2 = a$.

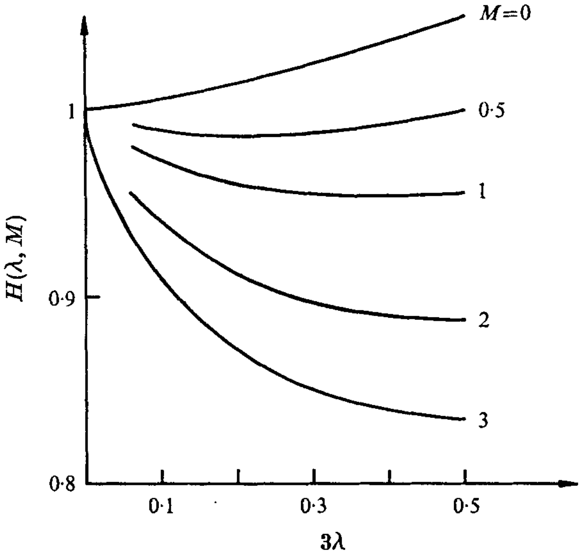

FIGURE 4. Function $H(\lambda, M)$, defined in equation (4.16).

In fact, straightforward algebra yields

$$\phi = a\zeta^2 - \frac{\zeta^3}{6} + a^2\frac{\zeta^5}{30} - a\frac{\zeta^6}{180} + \frac{\zeta^7}{2520} + \dots \quad (4.9)$$

This is now substituted into the boundary conditions (4.5) and (4.6), which then acquire the following form:

$$a\tilde{\zeta}^2 - \frac{\tilde{\zeta}^3}{6} + a^2\frac{\tilde{\zeta}^5}{30} - a\frac{\tilde{\zeta}^6}{180} + \frac{\tilde{\zeta}^7}{2520} + \dots = \lambda, \quad (4.10)$$

$$2a - \tilde{\zeta} + \tfrac{2}{3}a\tilde{\zeta}^3 - a\frac{\tilde{\zeta}^4}{6} + \frac{\tilde{\zeta}^5}{60} + \dots = \lambda\left(M - 2a\tilde{\zeta} + \frac{\tilde{\zeta}^2}{2} - a^2\frac{\tilde{\zeta}^4}{6} + a\frac{\tilde{\zeta}^5}{30} - \frac{\tilde{\zeta}^6}{360} + \dots\right). \quad (4.11)$$

These two equations can be solved by assuming the following expansions for a and $\tilde{\zeta}$:

$$a = (3\lambda)^{\frac{1}{3}}\{\tfrac{1}{2} + \alpha_1(3\lambda)^{\frac{2}{3}} + \alpha_2(3\lambda)^{\frac{4}{3}} + \dots\}, \quad (4.12)$$

$$\tilde{\zeta} = (3\lambda)^{\frac{1}{3}}\{1 + \beta_1(3\lambda)^{\frac{2}{3}} + \beta_2(3\lambda)^{\frac{4}{3}} + \dots\}. \quad (4.13)$$

Substitution of (4.12) and (4.13) into (4.10) and (4.11) and comparison of like powers of 3λ leads to the following result:

$$\tilde{\zeta} = (3\lambda)^{\frac{1}{3}}\left\{1 - \frac{M}{6}(3\lambda)^{\frac{2}{3}} + \left(\frac{M^2}{36} + \frac{107}{840}\right)(3\lambda)^{\frac{4}{3}} + \dots\right\}, \quad (4.14)$$

$$a\tilde{\zeta}^2 = \frac{3\lambda}{2}\left\{1 - \frac{M}{6}(3\lambda)^{\frac{2}{3}} + \left(\frac{M^2}{18} + \frac{97}{840}\right)(3\lambda)^{\frac{4}{3}} + \dots\right\}. \quad (4.15)$$

These expansions are correct up to the order explicitly written down.

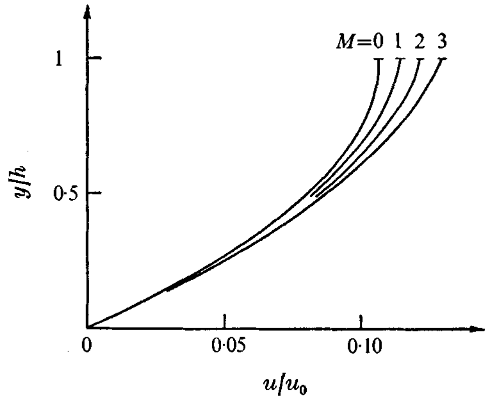

FIGURE 5. Velocity distribution parallel to the wall for $3\lambda = 0{\cdot}1$ and different values of Froude number M.

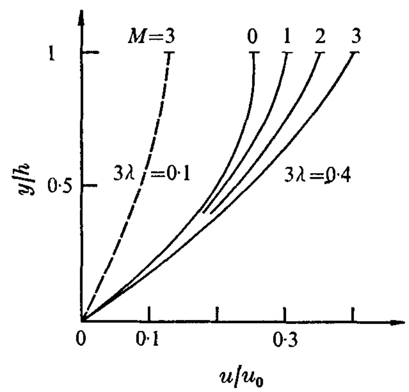

FIGURE 6. Velocity distribution for $3\lambda = 0{\cdot}4$ and different values of M (solid curves). For comparison the velocity for $3\lambda = 0{\cdot}1$ and $M = 3$ is also shown (dashed curve).

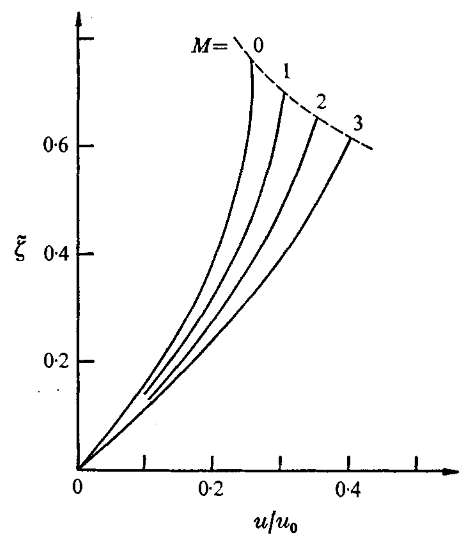

FIGURE 7. Velocity distribution for $3\lambda = 0{\cdot}4$ and different values of M (solid curves). Location of film surface is also shown (dashed curve).

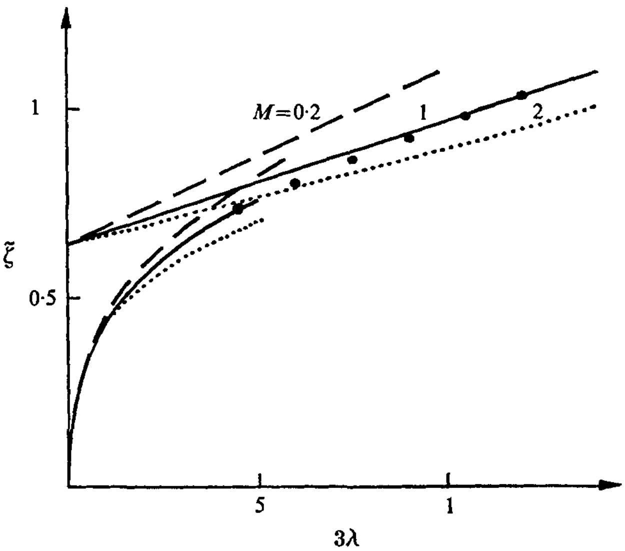

FIGURE 8. Dimensionless film thickness $\bar{\zeta}$ as a function of λ.
●, results of numerical calculations for $M = 1$.

The most important of these results is the expression (4.14) for the dimensionless film thickness. In dimensional terms this result is

$$h = (3R\nu\epsilon V/g^*)^{\frac{1}{3}}H(\lambda, M), \tag{4.16}$$

where $H(\lambda, M)$ is the function in curly brackets on the right-hand side of (4.14); see figure 4.

Figures 5–7 show velocity profiles for different values of λ and M. In figures 5 and 6 the ordinate ζ has been normalized with the dimensionless film thickness $\bar{\zeta}$, so that $\zeta/\bar{\zeta} = y/h = 1$ denotes the film surface for all velocity profiles. In figure 7 the dashed line connects the end points of the velocity profiles at the free surface, $\zeta = \bar{\zeta}$. Finally, figure 8 shows the dimensionless thickness of the film $\bar{\zeta}$ as a function of λ for three values of the Froude number M. The curved lines starting at the origin have been calculated from (4.13), whereas the straight lines are the results from (3.22). Obviously, these two results taken together provide already a good picture of the dependence of $\bar{\zeta}$ on λ within the whole range of λ from zero to infinity. Also shown in figure 8 are the results of purely numerical calculations for $M = 1$; these results are denoted by dots.

It is interesting, though rather obvious, to note that by restricting the expansions (4.8), (4.12) and (4.13) to the lowest-order term in λ respectively one obtains

$$\phi' = \bar{\zeta}\zeta - \tfrac{1}{2}\zeta^2. \tag{4.17}$$

This velocity profile is parabolic, and it satisfies the equation $\phi''' + 1 = 0$, which follows from (4.3) by neglect of the inertia terms on the left-hand side. Furthermore, (4.17) satisfies the boundary conditions (4.4) and (4.5), whereas the relation $\phi''(\bar{\zeta}) = 0$ is satisfied instead of (4.6). The latter relation is the condition for zero shear stress at the surface of the film! This shows that to a first approximation neither the inertia terms in the Navier–Stokes equations nor the shear stress induced by the momentum of the rain at the free surface play a role. In this approximation the flow is ordinary creeping flow with a parabolic profile of the velocity parallel to the wall. This velocity increases in the x direction in such a way (i.e. in proportion to x) that the total volume of rain falling onto the surface of the film is transported downstream in the film. Incidentally, the independence of the first-order solution of the shear stress induced at the free surface indicates that slight deviations of the direction of rainfall from the vertical have no effect on the first-order solution.

Surprisingly, the second-order solution (i.e. the first two terms in (4.15) and (4.16)) still satisfies the equation $\phi''' + 1 = 0$. Therefore, in the second-order approximation, only the shear stress induced at the film surface by the momentum of the rain has an influence on the solution, whereas the inertia terms in the Navier–Stokes equations are still without influence. Only in the third-order solution do the inertia terms play a role!

5. Applicability of the particular solution

As mentioned in the introduction, the problem treated here arose from a study of rain-water flow over a curved road surface (cf. Schleicher 1975, where much of the literature on experimental and theoretical studies based on hydraulic engineering type approximations is cited and surveyed). Therefore, the question arises as to what bearing on rain-water flow the *particular* solution (h independent of x!) discussed so far may possess. Of course, in order to apply that solution at all, one has to abstract from complicating details, like the effects of finite droplet size, surface roughness of the road, instability and wave formation, or even turbulence. The comparison of experimental and semi-theoretical results by Schleicher (1975), Shen & Li (1973) and Yoon & Wenzel (1971) seems to show that such effects are not very important in many rain flows of practical relevance. However, apart from these considerations, one has to acknowledge the fact that, for given ϵ and V, many more solutions of the Navier–Stokes equations are certainly possible which satisfy the boundary conditions of our problem, but which do not satisfy the assumptions (2.16) and (2.17) which led to the particular solution. Which of these (infinitely) many possible solutions is selected by nature? A tentative answer to this question will be provided by the discussion in this section. The discussion is based on an integral momentum method (see also Schleicher 1975; Yen & Wenzel 1970).

In order to derive the basic momentum equation (5.9) we simplify (2.2) by assuming $|u_{xx}| \ll |u_{yy}|$ (hydraulic boundary-layer approximation):

$$uu_x + vu_y = -p_x/\rho + g\sin\alpha + \nu u_{yy}. \tag{5.1}$$

Furthermore, in line with this simplification, a hydrostatic pressure distribution in the y direction is assumed:

$$p(x,y) = p_0(x) - \rho g(h-y)\cos\alpha. \tag{5.2}$$

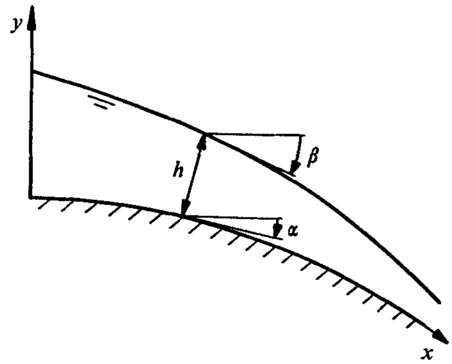

FIGURE 9. Notation for film flow with variable depth h.

Here, p_0 is again the pressure immediately below the surface of the film. Since in the general case, studied in this section, h is not independent of x, the surface inclination to the horizontal direction β (see figure 9) is not equal to α. Therefore, the pressure p_0 is given by the expression (2.11) for $-\sigma$, with β substituted for α (viscous normal stresses are neglected in boundary-layer approximation):

$$p_0 = \rho\epsilon(1-\epsilon)\, V^2\cos^2\beta. \tag{5.3}$$

With $|\alpha| \ll 1$ and $|dh/dx| \ll 1$, we may assume

$$\sin\beta \approx \sin\alpha - dh/dx, \tag{5.4}$$

$$\cos\beta \approx \cos\alpha \approx 1. \tag{5.5}$$

Then, a simple calculation shows that

$$-p_x/\rho + g\sin\alpha = g^*(\sin\alpha - dh/dx). \tag{5.6}$$

In (5.6) terms of the order h/R and h/R^* have been neglected; R^* denotes the radius of curvature of the surface of the film: $R^* = (d\beta/dx)^{-1}$. Equation (5.1) assumes the form

$$uu_x + vu_y = g^*(\sin\alpha - dh/dx) + \nu u_{yy}. \tag{5.7}$$

By integrating this equation over y from 0 to h, and by taking account of the equation of global continuity (equivalent to (2.27)), namely

$$\int_0^h u\,dy = \epsilon V\int_0^x \cos\alpha\,dx \approx \epsilon Vx, \tag{5.8}$$

the following momentum equation is derived:

$$\frac{d}{dx}\int_0^h u^2dy = g^*h\left(\sin\alpha - \frac{dh}{dx}\right) + \epsilon Vu(h) + \nu\frac{\partial u}{\partial y}\bigg|_0^h. \tag{5.9}$$

From (5.9) a differential equation for the film thickness $h(x)$ is deduced by assuming the velocity profile $u(x, y)$ to be of the form

$$u(x, y) = (3\epsilon Vx/h)\,(y/h - y^2/2h^2). \tag{5.10}$$

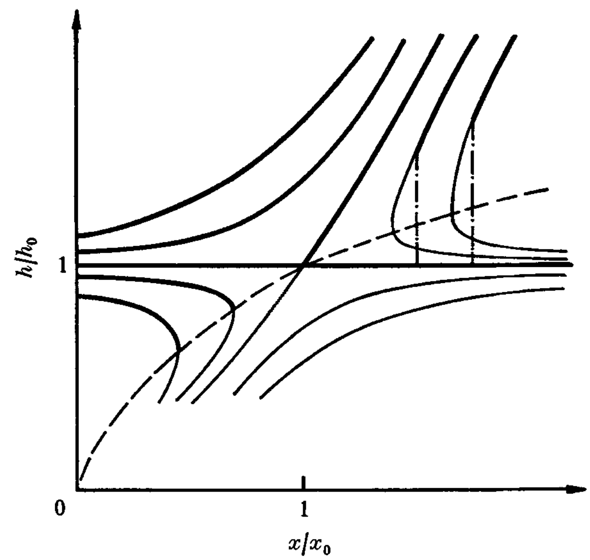

FIGURE 10. Solutions of equation (5.12) in the x, h plane (schematic). – – –, critical curve according to equation (5.15); —·—, hydraulic jump.

This profile, which is identical to (4.17), satisfies the no-slip condition at the wall and the condition of global continuity, relation (5.8). It does not satisfy the shear-stress condition at the free surface. Indeed, at the free surface, (5.10) gives $\partial u/\partial y = 0$, which is equivalent to zero shear stress. However, since it was shown in § 4 that the free-surface condition affects the solution only for values of λ not too small, (5.10) may be regarded as an acceptable approximation at least for small values of λ. In order to simplify the calculations we furthermore assume that

$$\epsilon Vh/\nu \ll 1 \tag{5.11}$$

and neglect terms of order $\epsilon Vh/\nu$ against 1. Substitution of (5.10) into (5.9) then yields

$$dh/dx = \sin\alpha\,(1-h_0^3/h^3)/(1-h_0^3x^2/h^3x_0^2), \tag{5.12}$$

where
$$h_0 = (3\nu\epsilon VR/g^*)^{\frac{1}{3}}, \tag{5.13}$$

and
$$x_0 = (5R\nu/2\epsilon V)^{\frac{1}{2}}. \tag{5.14}$$

As comparison with (4.16) shows, h_0 is the film thickness, in the approximation valid for small values of λ, for the particular solution discussed previously. Obviously, $h \equiv h_0$ is a particular solution of (5.12), as was to be expected.

The complete set of solutions of (5.12) can be discussed easily in a qualitative way by first noting that $x = x_0$, $h = h_0$ is a singular point in the x, h plane (figure 10). It is a saddle point with two solutions passing through the point, one of them being the particular solution $h \equiv h_0$. The denominator of the right-hand side of (5.12) vanishes on the curve

$$h/h_0 = (x/x_0)^{\frac{2}{3}}. \tag{5.15}$$

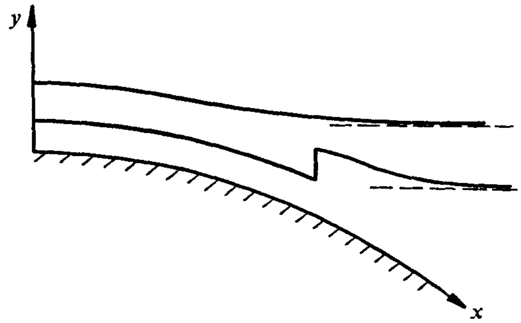

FIGURE 11. Solutions of equation (5.12) with horizontal asymptote of free surface (schematic).

This curve is shown in figure 10 as the dotted line. A brief calculation shows that vanishing of the denominator in (5.12) is equivalent to

$$\int_0^h u^2 dy/(g^*h^2) = 1. \tag{5.16}$$

The left-hand side of (5.16) is the square of an effective local Froude number, defined with the local film thickness as the characteristic length and the mean-square velocity in the film as the characteristic velocity. Taking into account the different signs of numerator and denominator in the different areas of the x, h plane bounded by the curve (5.15) and the solution $h \equiv h_0$, one can easily confirm the qualitative nature of the solutions of (5.12), as shown in figure 10.

If the road has a breadth l from its centre to its edge, larger than x_0, and if the flow is not obstructed at the edge, the solution is everywhere given by the particular solution $h \equiv h_0$. The flow is subcritical (effective local Froude number < 1) for $x < x_0$ and supercritical for $x > x_0$. If $l < x_0$ and if the edge of the road is unobstructed, so that the water leaves the road in a free overfall at the edge, the local Froude number should reach the value 1 in the immediate vicinity of the edge (see Schleicher 1975). In that case a solution from the lower left-hand domain of the x, h plane will be realized, which reaches the critical value 1 of the local Froude number at the edge of the road (with a vertical tangent, which of course is unrealistic and due to the simplifying assumptions made in the hydraulic approximation). However, for realistic values of road curvature ($R = O(10^2\,\mathrm{m})$) and rain properties ($\epsilon = O(10^{-6})$, $V = O(10\,\mathrm{m/s})$), these solutions differ appreciably from the particular solution only in the immediate vicinity of the edge of the road, whereas they coincide nearly exactly with the particular solution over most of the breadth of the road. Therefore, the particular solution discussed in the previous chapters is for all practical purposes the solution also in that case.

Equation (5.12) shows that $dh/dx \rightarrow \sin\alpha$ as $h/h_0 \rightarrow \infty$. Therefore, in the upper half of the x, h plane the slope of all solutions tends to $\sin\alpha$, which means that the film surface becomes horizontal. This type of solution is sketched in figure 11. Also, solutions with a hydraulic jump at a position $x > x_0$ are possible. Such a jump is indicated in figure 10 by a dash–dot line, and the physical properties of the solution should be clear from figure 11. For a further discussion the reader should consult the paper by Schleicher (1975). Though this paper is concerned with the case of a road with constant inclination α ('roof-top'), the manifold of solutions is quite similar to the one studied here.

Concluding this section, we note that the remarks made at the end of §4 about the effects of the inertia terms in the Navier–Stokes equations and the shear stress induced at the film surface show the way to a simple first-order approximation for rain-water films. This approximation is valid also if the curvature of the wall is not constant. The approximation is based on the equation

$$\nu u_{yy} + g \sin\alpha = 0, \tag{5.17}$$

which is derived from (5.7) by neglecting the inertia terms and the term dh/dx and by identifying g^* with g, which is justified for natural rain, for which $\epsilon \ll 1$. Integrating (5.17) twice with respect to y, applying the no-slip condition ($u(0) = 0$) and the condition of vanishing shear stress at the film surface ($\partial u/\partial y = 0$ for $y = h$), one obtains

$$u = (g \sin\alpha/\nu)(hy - \tfrac{1}{2}y^2). \tag{5.18}$$

Inserting this result into the exact form of the global continuity condition (5.8) finally leads to

$$h^3 = (3\epsilon\nu V/g \sin\alpha) \int_0^x \cos\alpha \, dx. \tag{5.19}$$

For $\sin\alpha = x/R$ and $\cos\alpha \approx 1$ the result $h \equiv h_0$ (with $g^* = g$) is recovered from (5.19); in that case (5.18) reduces to (5.10). However, (5.19) is valid for general variation of α with x, even if α no longer satisfies the condition $|\alpha| \ll 1$! Of course, the preceding discussion makes it clear that the approximations made have led to a particular solution only. For other possible solutions, as sketched in figure 11, neither dh/dx nor the inertia terms may be neglected.

6. Limitations imposed by the neglect of curvature terms

In order to get some feeling for the orders of magnitude of different quantities discussed in the previous sections, some numbers for a typical example of the flow of rain-water over a road surface may be useful. The radius of curvature of the road is taken as $R = 50$ m. For moderate to heavy rainfall the assumptions $\epsilon = 10^{-6}$ and $V = 5$ m/s are appropriate. The kinematic viscosity of water is $\nu = 10^{-6}$ m^2/s and the effective constant of gravity is $g^* \approx g = 10$ m/s^2. These values give $M = 0{\cdot}224$, $Re = 1{\cdot}12 \times 10^9$ and $\lambda = 0{\cdot}0075$; obviously, the theory of §4 is applicable, and the depth of the film is given, to a very good approximation, by $h_0 = 0{\cdot}42 \times 10^{-3}$ m; furthermore, $x_0 = 5$ m. The neglected curvature terms are of the order of $h_0/R = 8{\cdot}4 \times 10^{-6}$. The parameter $\epsilon V h/\nu$ has the value $2{\cdot}1 \times 10^{-3}$.

Generally, in the domain of validity of the theory presented in §4, i.e. for $\lambda \ll 1$, curvature terms should be negligible as long as $h/R = \xi Re^{-\frac{1}{2}} \ll 1$ [cf.

(4.7)]. An approximation to ξ is given by $(3\lambda)^{\frac{1}{3}}$ [cf. (4.13)]. Therefore, neglect of curvature terms is justified if $(3\lambda)^{\frac{1}{3}}Re^{-\frac{1}{2}} \ll 1$, or

$$\lambda \ll Re^{\frac{3}{2}}. \tag{6.1}$$

Since $\lambda \ll 1$, this condition is satisfied if $Re \gtrsim 1$.

In the domain of validity of the theory of § 3, i.e. for $\lambda \gg 1$, a first approximation to $h/R = \tilde{h}$ is given by $\tilde{h}_0$ [cf. (3.15)]. Figure 9 shows that in a rough approximation for $M \lesssim 3$ one can assume $\tilde{h}_0 \approx \epsilon M$. Therefore, the neglect of curvature terms is now justified if

$$\epsilon M \ll 1. \tag{6.2}$$

Neglect of the curvature term of order $\epsilon^2 M^2$ compared with the second term on the right-hand side of (3.22) is justified if $\epsilon^2 M^2 \ll Re^{-\frac{1}{2}}$, or

$$\epsilon M \lambda \ll 1. \tag{6.3}$$

Therefore, validity of (3.22) is guaranteed only if besides $\lambda \gg 1$, the relation (6.3) is satisfied. This means that ϵM has to be smaller by at least an order of magnitude than $1/\lambda$. For example, in the numerical example mentioned above, increasing ϵ from 10^{-6} to 5×10^{-3} (which is unrealistic for natural rain) without changing V, R and ν increases λ to 37·5 without changing M and Re. Then the conditions $\lambda \gg 1$ and $\epsilon M \lambda \ll 1$ are satisfied simultaneously.

7. Concluding remarks

The problem treated in the previous sections admits a number of generalizations and corollaries. For example, the theory can be applied nearly without change to flow over a spherical surface with radius R. The governing equations, in that case, are those for rotationally symmetric stagnation-point flow with appropriate boundary conditions. Instead of the result (3.15) one now has the simple formula

$$\tilde{h}_0 = \epsilon M/(1+M), \tag{7.1}$$

and instead of (4.16),

$$h_0 = (3R\nu\epsilon V/2g^*)^{\frac{1}{3}}\hat{H}(\lambda, M), \tag{7.2}$$

with $\hat{H}(0, M) = 1$.

A possible generalization of rain flow over a cylindrical road surface with horizontal axis is the extension to the case of a road which is inclined to the horizontal direction (an ascending or descending road). The results obtained previously are still valid for the transverse flow of rain-water. In addition to that flow, a flow in direction of the road itself is established, which is driven by gravity and is governed by simple linear equations. Because of this linearity it is easy to calculate that flow. A further complication which is easily taken account of is obliqueness of the rainfall with respect to the direction of the axis of the cylindrical road. Also in that case our previous results for the transverse flow remain unchanged, and an additional flow in the direction of the road is set up. The additional flow is now driven by the momentum of the rain in the direction of the axis of the road. Again, the additional flow is governed by linear equations, and hence is easily calculated.

Finally, it should be mentioned that the rain-flow problem has some similarities to the flow generated by film condensation (Beckett & Poots 1975; Cheng 1961). Here, the condensing vapour is a source of liquid at the surface of the film. In the simplest case the film is again driven by gravity. However, different boundary conditions at the film surface preclude too close an analogy with the rain flow studied here.

A last remark is in order. As already mentioned in §5, real rain-water films differ in many respects from the idealized solution studied here. Particularly, finite drop size might have some effects worthy of study, including premature onset of turbulence (see Yoon & Wenzel 1971). Yet, a prerequisite for a quantitative assessment of such effects is the availability of a simple idealized but exact solution of the basic equations which can be compared with experimental results. Previous efforts in that direction suffer from a lack of such solutions and are based on approximate methods or global momentum considerations. The particular solution found above provides a simple solution that avoids approximations as far as possible. This, it is hoped, justifies its presentation.

REFERENCES

BECKER, E. 1975 *Mech. Res. Comm.* **2**, 149–154.

BECKETT, P. M. & POOTS, G. 1975 *Mech. Res. Comm.* **2**, 61–66.

BÖHME, G. & BECKER, E. 1972 *J. Appl. Math. Phys.* (*Z. angew. Math. Phys.*) **23**, 983–990.

CHENG, M. M. 1961 *A.S.M.E. J. Heat Transfer*, **83**, 55–60.

SCHLEICHER, U. 1975 Regenwasserfilme auf schwach geneigten Flächen. Dissertation, Darmstadt D 17.

SCHLICHTING, H. 1965 *Grenzschichttheorie*, 5th edn, Karlsruhe: G. Braun.

SHEN, H. W. & LI, R. M. 1973 *A.S.C.E. J. Hydr. Div.* **99**, 771–791.

YEN, B. C. & WENZEL, H. G. 1970 *A.S.C.E. J. Hydr. Div.* **96**, 801–814.

YOON, Y. N. & WENZEL, H. G. 1971 *A.S.C.E. J. Hydr. Div.* **97**, 1367–1386.